Accurate variational approach for vector solitary waves

C. Paré*
Département de Physique, Equipe Laser et Optique Guidée, Centre d'Optique, Photonique et Laser, Université Laval, Sainte-Foy, Québec, Canada G1K 7P4

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We describe how the variational method can be used to derive accurate analytical approximations of the solitary wave solutions of coupled nonlinear Schrödinger equations. This can be achieved with a hyperbolic secant ansatz combined with a Taylor series expansion of the time-averaged interaction Lagrangian. This technique is illustrated here in the context of third-order nonlinear optics but should prove also useful for other nonlinear systems. [S1063-651X(96)04906-9]

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I. INTRODUCTION

Considering the fact that realistic optical fibers are not truly single mode, owing to the presence of birefringence, Menyuk [1] has pointed out that a correct description of nonlinear pulse propagation in a fiber should be based on the so-called vector nonlinear Schrödinger (NLS) equation. In conventional soliton units, the propagation in the anomalous dispersion regime is then modeled by a pair of coupled NLS equations [1]:

$$\begin{aligned} &i\left(\frac{\partial\psi_1}{\partial z}+\delta\frac{\partial\psi_1}{\partial\tau}\right)+\frac{1}{2}\frac{\partial^2\psi_1}{\partial\tau^2}+(|\psi_1|^2+\sigma|\psi_2|^2)\psi_1\\ &\qquad+\mu\psi_2^2\psi_1^*\exp(-iR\delta z)=0,\\ &i\left(\frac{\partial\psi_2}{\partial z}-\delta\frac{\partial\psi_2}{\partial\tau}\right)+\frac{1}{2}\frac{\partial^2\psi_2}{\partial\tau^2}+(|\psi_2|^2+\sigma|\psi_1|^2)\psi_2\\ &\qquad+\mu\psi_1^2\psi_2^*\exp(+iR\delta z)=0,\end{aligned}\tag{1}$$

where ψ_1 and ψ_2 represent the normalized slowly varying envelopes of the slow and fast polarization components, respectively. The propagation distance z is expressed in units of dispersion length, δ is half of the inverse group-velocity mismatch due to the birefringence, τ is the normalized time in a reference frame moving at an average group velocity; the cross-phase modulation factor $\sigma=2/3$ and $\mu=1/3$ in this case. $R=(8\pi c/\lambda_0)(0.567\ T)$ where λ_0 is the wavelength and T is the full width at half maximum (FWHM) of the pulse intensity profile (for a sech^2 pulse shape).

The important point first demonstrated numerically by Menyuk [2,3], and later observed experimentally [4], is the fact that the nonlinearity can still be strong enough to simultaneously compensate for the dispersion and combat the tendency of the polarization components to split apart as a result of their group velocity difference. The two polarization pulses trap each other and form what is called a vector solitary wave. To investigate the nature of these pulses, the coupled NLS equations can be simplified by first introducing a phase transformation [5,6]

$$\begin{aligned}\psi_1&=U\exp[i(\delta^2z/2-\delta\tau)],\\ \psi_2&=V\exp[i(\delta^2z/2+\delta\tau)],\end{aligned}\tag{2}$$

and by neglecting the rapidly oscillating last term on the left-hand side of each equation, as is justified for pulses in the picosecond range (see [1]). The equations then read as

$$\begin{aligned} &i\,\frac{\partial U}{\partial z}+\frac{1}{2}\frac{\partial^2U}{\partial\tau^2}+(|U|^2+\sigma|V|^2)U=0,\\ &i\,\frac{\partial V}{\partial z}+\frac{1}{2}\frac{\partial^2V}{\partial\tau^2}+(|V|^2+\sigma|U|^2)V=0.\end{aligned}\tag{3}$$

The phase transformation introduced above is indicative of the physical process giving rise to the mutual trapping of the polarization components: through cross-phase modulation, each partial pulse shifts its central frequency in such a way that both group velocities become equal [2–4].

It turns out that the same evolution equations [Eqs. (3)] also govern an important limit case, relevant to the operation of soliton fiber lasers, where the induced *nonlinear* birefringence is thought to be the dominant mechanism so that the fiber can be considered practically *linearly* isotropic (δ=0) [7–11]. Indeed, in such a case, the evolution equations (1) can be transformed into Eqs. (3) (with σ now equal to 2) if the field is rather expressed in terms of *circularly* polarized components [7–11]. Allowing for polarization rotation as the pulses propagate along the fiber, vector solitary waves of the form

$$\begin{aligned}U(z,\tau)&=u(\tau)\exp[i\varphi z],\\ V(z,\tau)&=v(\tau)\exp[i\beta z],\end{aligned}\tag{4}$$

where u and v are real functions, have been analyzed recently [7–11]. The pulse shapes of these solitary waves must then satisfy the following coupled ordinary differential equations:

$$\begin{aligned}&\tfrac{1}{2}\ddot{u}+(u^2+\sigma v^2)u-\varphi u=0,\\ &\tfrac{1}{2}\ddot{v}+(v^2+\sigma u^2)v-\beta v=0.\end{aligned}\tag{5}$$

The phase mismatch $(\beta-\varphi)$ introduces a polarization rotation which might play an important role in the pulse formation in soliton fiber lasers.

*FAX: (418)-656-2623. Electronic address: cpare@phy.ulaval.ca

As a third example in nonlinear optics where the same coupled equations [Eqs. (5)] appear, let us mention the case of mutual trapping of counterpropagating beams (*spatial* solitary waves [12]) which might represent the ultimate state of a pattern formation initiated by a transverse modulational instability [13,14] (in that case, $\sigma=1$ or 2, depending on whether or not spatial diffusion washes out the induced grating).

Except for a few particular cases discussed further below, Eqs. (5) must generally be solved numerically (e.g., by the shooting method [15]). Considering the widespread occurrence of these vector solitary waves, it would be interesting to find approximate analytical solutions of the governing coupled equations. In this paper, we describe how the variational method (also called Whitham's averaging method [16]) can provide simple and accurate analytical expressions for the pulses' (or beams') profiles. Although Eqs. (5) possess different types of solutions [7–10], we limit ourselves to the fundamental (i.e., without nodes) solitary waves. This restriction is further justified by the fact that the higher-order solutions have been found unstable in some instances [7] and the interest for developing accurate approximations in such a case is reduced.

To limit the number of parameters, it is worth pointing out that the solitary wave equations [Eqs. (5)] can be renormalized in terms of the phase parameter φ [9,10,12]: through the change of variables $x\rightarrow x/\varphi^{1/2}$, $u\rightarrow u\varphi^{1/2}$, $\nu\rightarrow\nu\varphi^{1/2}$, $\beta\rightarrow\beta/\varphi$, one recovers Eqs. (5) with φ now equal to 1. In the following, we then pose $\varphi=1$, having this scaling in mind for $\varphi\neq1$. The parameter space can also be limited to the region $\beta\geqslant\varphi$. Indeed, the above scaling invariance implies that the results for $\beta\leqslant\varphi$ can be inferred from those obtained for $\beta\geqslant\varphi$ by using the symmetry

$$u,\nu(\beta^{1/2}x)\big|_{\beta/\varphi\leqslant1}=\nu,u(\varphi^{1/2}x)\big|_{(\beta/\varphi)'=\varphi/\beta\geqslant1}.$$

This also implies the energy scaling:

$$E_{u,\nu}\big|_{\beta/\varphi\leqslant1}=\frac{\beta}{\varphi}\,E_{\nu,u}\bigg|_{(\beta/\varphi)'=\varphi/\beta\geqslant1}.$$

II. VARIATIONAL APPROACH

The first step in the variational approach (see [17] for an introduction to this method when applied to the single NLS equation) consists in finding the Lagrangian density associated with the governing equations. In the present case, it can be shown that the coupled Eqs. (5) can be derived (via the Euler-Lagrange equations) from the following variational problem:

$$\delta L\equiv\delta\int\mathscr{L}\,d\tau=0,$$

where the Lagrangian density is given by

$$\mathscr{L}=u_\tau^2+\nu_\tau^2+2\varphi u^2+2\beta\nu^2-(u^4+\nu^4+2\sigma u^2\nu^2). \tag{6}$$

This formulation is then exploited as the basis of a Rayleigh-Ritz optimization procedure. The crucial step is the choice of an appropriate ansatz for the trial solutions. As mentioned above, the set of coupled Eq. (5) possesses exact analytical solutions in some particular cases, and this can be a useful guide. For example, if one of the components is absent, the other one then satisfies the single NLS equation. Hence, $\nu=0$, $u=\sqrt{2\varphi}\mathrm{sech}(\sqrt{2\varphi}\tau)$ or $u=0$, $\nu=\sqrt{2\beta}\mathrm{sech}(\sqrt{2\beta}\tau)$ are solutions of the system (5). As another example, $u=\pm\nu=\sqrt{2\varphi/(\sigma+1)}\mathrm{sech}(\sqrt{2\varphi}\tau)$ also solves Eqs. (5). Let us also mention that a perturbative analysis of the system (5) has also been carried out [7,9,10,12] around the solution $\nu=0$, $u=\sqrt{2\varphi}\mathrm{sech}(\sqrt{2\varphi}\tau)$, in the limit $|\nu|\ll|u|$. (For other particular solutions of the coupled NLS equations, but less relevant to the present analysis, see [18–21]). Inspired from these cases, we use the following ansatz for u and v:

$$\begin{aligned}u&=a_1\mathrm{sech}(b_1\tau),\\ \nu&=a_2\mathrm{sech}(b_2\tau).\end{aligned}\tag{7}$$

This ansatz is also justified by the fact that it represents an exact solution (with $b_1=b_2$) in the case $\sigma=1$ (Manakov's system [18]). The Euler-Lagrange equations, applied to the integrated Lagrangian density (averaged Lagrangian L) then provide the "best" choice for the unknown parameters $a_{1,2}$ and $b_{1,2}$. Unfortunately, when $b_1\neq b_2$ (the general case), the interaction part of the Lagrangian density ($\mathscr{L}_{\mathrm{int}}=-2\sigma u^2\nu^2$) cannot be integrated analytically. This difficulty is usually circumvented by the choice of a different ansatz. In particular, a *Gaussian* ansatz ($\sim a_i\exp[-b_it^2]$, $i=1$ or 2) seems a natural choice since it allows for an analytical integration [5]. The results with a *Gaussian* trial function will not be detailed here (see [5]) but will be illustrated graphically for the sake of comparison. Its main limitation is a poor description of the wings of the profiles and this can be understood from the asymptotic limit ($|\tau|\rightarrow\infty$) of the system (5) [9]. In that region, one can drop the nonlinear terms and then conclude that both profiles must present an *exponential* (rather than Gaussian) tail, as also evidenced by the particular analytical solutions discussed above.

Here, we show how one can maintain the (expected) more accurate sech trial functions (7) without penalty, in terms of numerical effort. We simply notice that in most cases, inspection of the numerical solutions reveals that the pulse width ratio b_1/b_2 is in the range $0.7\approx\leqslant b_1/b_2\leqslant\approx1.3$ and we then suggest to develop the troublesome integral in a Taylor series around the point $b_1=b_2$. As clearly demonstrated below, the series turns out to be rapidly converging.

To proceed, using the ansatz (7), the Euler-Lagrange equations applied to the averaged Lagrangian lead to the following nonlinear system of four coupled equations for the unknown parameters:

$$\frac{b_1}{3}+\frac{2\varphi}{b_1}-\frac{2}{3}E_1-\sigma E_2P(\eta)=0, \tag{8a}$$

$$\frac{b_2}{3}+\frac{2\beta}{b_2}-\frac{2}{3}E_2-\sigma E_1P(1/\eta)=0, \tag{8b}$$

$$\frac{b_1}{3}-\frac{2\varphi}{b_1}+\frac{1}{3}E_1+\sigma E_2\frac{1}{\eta}T(1/\eta)=0, \tag{8c}$$

$$\frac{b_2}{3}-\frac{2\beta}{b_2}+\frac{1}{3}E_2+\sigma E_1\eta T(\eta)=0, \tag{8d}$$

where we introduce the width ratio η:

$$\eta=\frac{b_1}{b_2},$$

the pulses' energies $E_{1,2}$ ($E_u=E_1$ and $E_\nu=E_2$):

$$E_{1,2}=\int_{-\infty}^{+\infty}a_{1,2}^2\mathrm{sech}^2\, b_{1,2}\tau\, d\tau=2\frac{a_{1,2}^2}{b_{1,2}}$$

and the integrals $P(\eta)$ and $T(\eta)$:

$$P(\eta)=\frac{1}{2}\int_{-\infty}^{+\infty}\mathrm{sech}^2 x\ \mathrm{sech}^2\,\eta x\ dx, \tag{9a}$$

$$T(\eta)=\int_{-\infty}^{+\infty}x\ \mathrm{sech}^2 x\, th x\ \mathrm{sech}^2\,\eta x\ dx. \tag{9b}$$

The integral $T(\eta)$ arises from a term of the form $\partial P/\partial\eta$. We note parenthetically that it is straightforward to show that the particular exact analytical solutions discussed above can be recovered from this system. This is obviously not the case when Gaussian trial functions are used.

It is important to note that the perturbative analysis briefly mentioned above [before Eq. (7)] reveals that the fundamental solutions we are interested in exist only in the range $1/\beta_c\leqslant\beta\leqslant\beta_c$, where $\beta_c=2.438$ (for $\sigma=2$) corresponds to the value at which $E_\nu\to 0$. We now solve the nonlinear system (8). First, straightforward algebra yields

$$\frac{E_2}{E_1}=\frac{1-3\eta\sigma[P(1/\eta)-\eta T(\eta)]}{\eta-3\sigma\left[P(\eta)-\dfrac{1}{\eta}T(1/\eta)\right]}. \tag{10}$$

If β is considered as the free parameter, then the unknown value of η has to be determined by solving (e.g., by using the bisection method) the implicit equation

$$\beta\eta=\frac{\dfrac{E_2}{E_1}+\sigma[P(1/\eta)+\eta T(\eta)]}{1+\sigma\dfrac{E_2}{E_1}\left[P(\eta)+\dfrac{1}{\eta}T(1/\eta)\right]}. \tag{11}$$

A similar implicit equation must also be solved with the Gaussian approximation. More simply, Eqs. (10) and (11) are better seen as a *parametric solution* $E_{1,2}$ vs β, where the parametric variable η is allowed to vary in the range for which $1/\beta_c\leqslant\beta\leqslant\beta_c$. E_1 and E_2 can now be found from

$$E_1^2=\frac{8}{\left(1+\sigma\dfrac{E_2}{E_1}\left[P(\eta)+\dfrac{1}{\eta}T(1/\eta)\right]\right)\left(1+3\sigma\dfrac{E_2}{E_1}\left[P(\eta)-\dfrac{1}{\eta}T(1/\eta)\right]\right)} \tag{12}$$

and Eq. (10), and the other parameters are then determined from

$$b_1=\frac{E_1}{2}+\frac{3}{2}\sigma E_2\left[P(\eta)-\frac{1}{\eta}T(1/\eta)\right], \tag{13}$$

$$b_2=\frac{b_1}{\eta}, \tag{14}$$

and

$$a_{1,2}=\sqrt{b_{1,2}E_{1,2}/2}. \tag{15}$$

The solution above can be fully analytical if one makes a Taylor expansion in η of the integrals $P(\eta)$ and $T(\eta)$. It is easy to demonstrate that, to the second order in $(1-\eta)$, one has

$$P(\eta)\cong 1-\frac{\eta}{3}+\frac{1}{3}\left[1-\frac{\pi^2}{15}\right](1-\eta)^2, \tag{16}$$

and

$$T(\eta)\cong\frac{1}{3}-\frac{2\pi^2}{45}(\eta-1)+\frac{\pi^2}{30}(\eta-1)^2. \tag{17}$$

The validity of these second-order expansions is demonstrated graphically in Fig. 1. In the next section, the analytical solution just derived is compared with the exact numerical solution as well as with the Gaussian approximation.

III. NUMERICAL RESULTS

In this section, we demonstrate how accurate this simple approach can be. To illustrate this, we restrict ourselves to the case $\sigma=2$ which is of relevance for the problem of polarization rotation in isotropic fibers and also for the problem of counterpropagating beams in a Kerr-like medium. Figure 2(a) first compares the dispersion curves (energy vs propagation constant) obtained numerically (solid lines) by using the shooting method [15] with the variational results based on a Gaussian ansatz (dashed lines with circles). The curve (E_u+E_ν) is relevant to the stability analysis for the counterpropagating problem, as will be discussed elsewhere. Although the agreement is relatively good, one can appreciate [Fig. 2(b)] the great improvement obtained with the sech ansatz combined with a second-order expansion of the integrals P and T. The agreement is then almost perfect (actually, a first-order expansion is almost as good). Figure 3 displays the variation of the width ratio as a function of β, as determined from Eq. (11). Its range of variation justifies the second-order expansion proposed here (see Fig. 1).

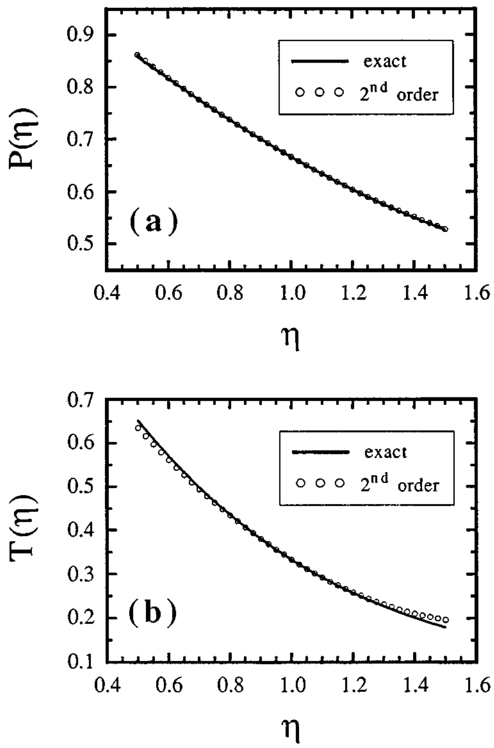

FIG. 1. Variation of the integrals $P(\eta)$ and $T(\eta)$ [Eqs. (9a),(9b)] as a function of the width ratio η. A comparison is made between the exact numerical integration (solid lines) and the second-order analytical approximations (circles) [Eqs. (16),(17)].

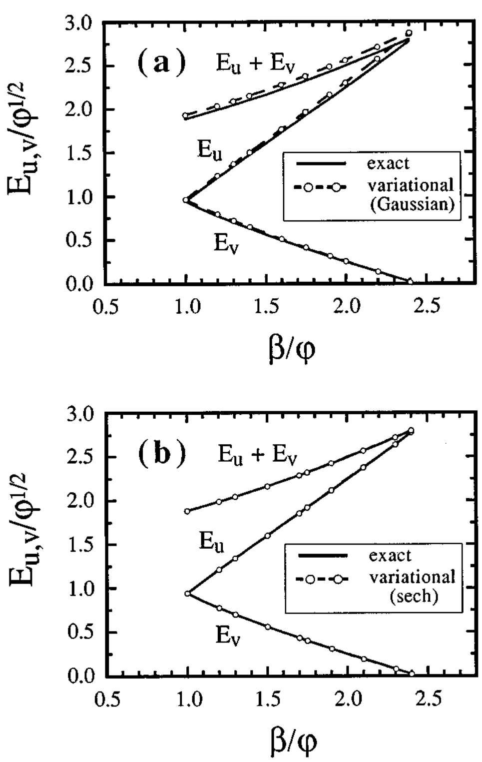

FIG. 2. Comparison between the numerically obtained dispersion curves (solid lines) with the variational approximations using (a) a Gaussian ansatz; (b) a sech ansatz. The parameter σ is fixed equal to 2. The results for $\beta \leqslant \varphi$ can be inferred from symmetry properties, as discussed in the text.

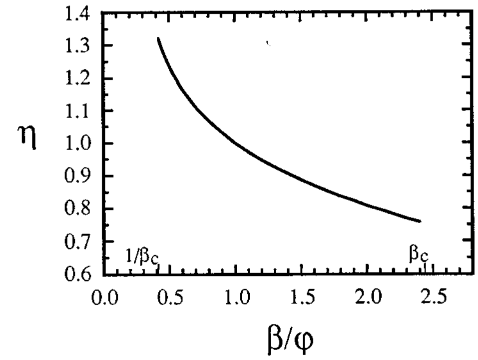

FIG. 3. Variation of the width ratio parameter η as a function of the propagation constant β. The parameter $\sigma=2$.

The dispersion curves depict a *gross* characteristic of the solutions which is generally less sensitive to the detailed shapes of the solitary wave profiles. To demonstrate this, Fig. 4 now compares the exact (numerically obtained) pro-

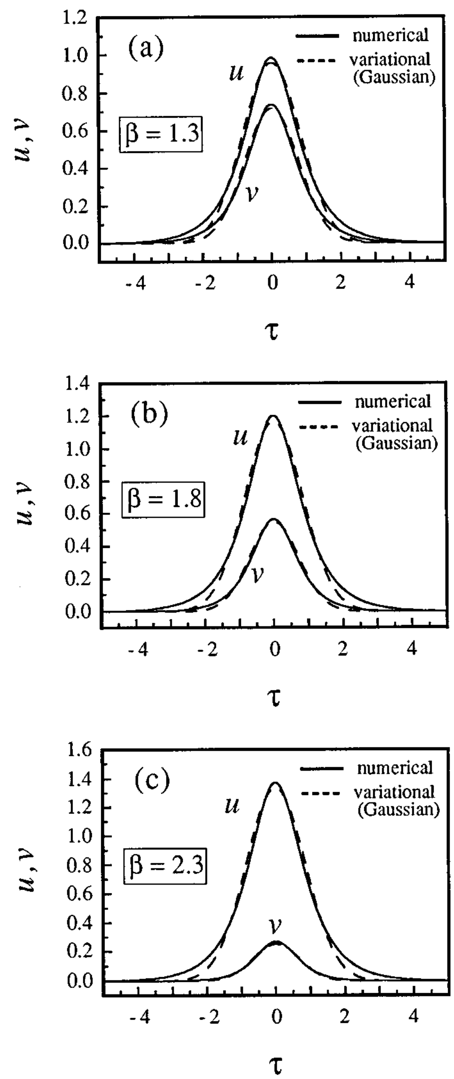

FIG. 4. Comparison between the numerically obtained solitary wave profiles u and v and the predictions of the variational model using a Gaussian ansatz. (a) $\beta=1.3$; (b) $\beta=1.8$; (c) $\beta=2.3$. The parameter σ is equal to 2.

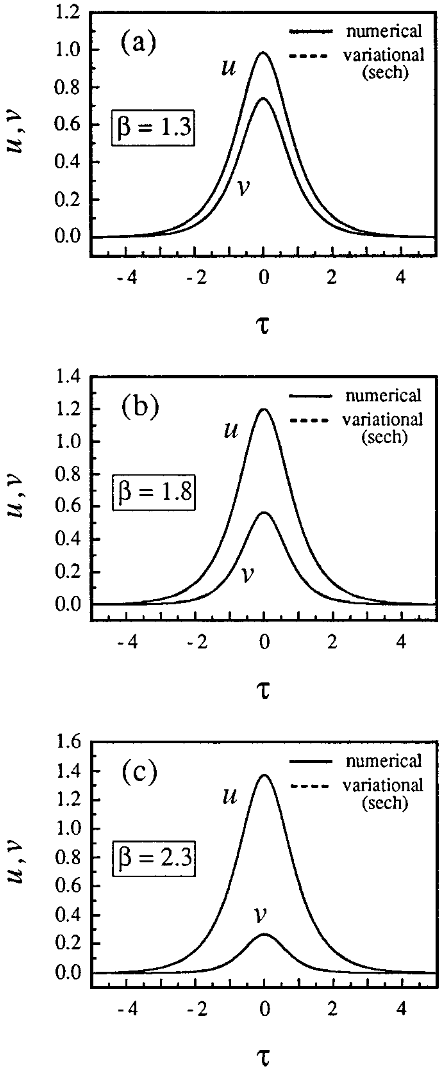

FIG. 5. Comparison between the numerically obtained solitary wave profiles u and v and the predictions of the variational model using a sech ansatz. (a) β=1.3; (b) β=1.8; (c) β=2.3. The parameter σ is fixed equal to 2. These results should be compared with those of Fig. 4.

files u and ν with the Gaussian variational approximation for three different values of the parameter β. The approximate solution provides a relatively good estimate of the width and amplitude parameters but fails to describe properly the wings of the pulses. In sharp contrast, Fig. 5 shows that the sech trial solution is nearly exact and this, without additional numerical work. In terms of polarization rotation, such accurate (and still simple) analytical results can be useful for determining the length of polarization switch ($L_S=[\pi/|\varphi-\beta|]$) in terms of the energy ratio E_ν/E_u [11]. This is illustrated in Fig. 6.

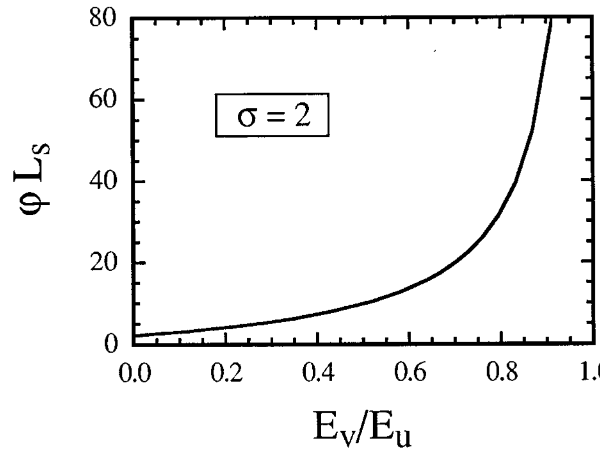

FIG. 6. Length of polarization switch vs the energy ratio of the polarization components as predicted from the variational model using a sech ansatz. The parameter σ is fixed equal to 2.

IV. DISCUSSION

This work dealt with *stationary* solutions of the vector NLS equations. Regarding the important question of stability of these solutions, we must emphasize that even though the solitary waves described here are relevant to various nonlinear problems, their stability must be analyzed separately for each situation, since the *dynamical* equations from which they originate are different. We are currently investigating this point in the context of counterpropagating solitary waves (spatial domain) (see [2,3,22] for the case of birefringent fibers and [7,10] for the case of isotropic fibers) and the present results can provide a good basis of analysis. This will be detailed elsewhere.

We believe that the idea of a Taylor series expansion of the integral of the interaction Lagrangian, as proposed here, can also prove fruitful in other problems. Hence we are also currently investigating the case of solitary waves in a quadratic ($\chi^{(2)}$) medium, a problem receiving a growing attention in the literature. The important problem of soliton interaction in a nonlinear directional coupler has already been considered with a variational approach under simplifying assumptions [23,24]. It might be worth revisiting this problem along the lines suggested in this paper.

Besides its mathematical interest, the result of this work might be useful for practical design or modeling considerations. This would be the case, for example, for systems involving polarization rotation such as soliton fiber lasers. Finally, we would like to mention the work of Bhakta [25] concerning another approach for approximating the solution of the coupled NLS equations. In that work, the Hirota method is used in a perturbative analysis of the *z-dependent* evolution of interacting solitary waves. We believe, however, that the variational approach suggested here is more appropriate for the description of fundamental *stationary* solitary waves, at least in the context of the three examples discussed here and borrowed from nonlinear optics.

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[1] C. R. Menyuk, IEEE J. Quantum Electron. **23**, 174 (1987).
[2] C. R. Menyuk, Opt. Lett. **12**, 614 (1987).
[3] C. R. Menyuk, J. Opt. Soc. Am. **5**, 392 (1988).
[4] M. Islam, C. D. Poole, and J. P. Gordon, Opt. Lett. **14**, 1011 (1989).
[5] D. J. Kaup, B. A. Malomed, and R. S. Tasgal, Phys. Rev. E **48**, 3049 (1993).
[6] G. P. Agrawal, *Nonlinear Fiber Optics*, 2nd ed. (Academic, San Diego, 1995), Chap. 7 and references therein.
[7] Y. Silberberg and Y. Barad, Opt. Lett. **20**, 246 (1995).
[8] M. Haelterman, A. P. Sheppard, and A. W. Snyder, Opt. Lett. **18**, 1406 (1993).
[9] M. Haelterman and A. P. Sheppard, Phys. Rev. E **49**, 3376 (1994).
[10] M. Haelterman and A. P. Sheppard, Phys. Lett. A **194**, 191 (1994).
[11] V. V. Afanasjev, Opt. Lett. **20**, 270 (1995).
[12] M. Haelterman, A. P. Sheppard, and A. W. Snyder, Opt. Commun. **103**, 145 (1993).
[13] W. J. Firth and C. Paré, Opt. Lett. **13**, 1096 (1988).
[14] G. Grynberg, Opt. Commun. **66**, 321 (1988).
[15] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes: The Art of Scientific Computing* (Cambridge University Press, Cambridge, 1989).
[16] G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974).
[17] D. Anderson, Phys. Rev. A **27**, 3135 (1983).
[18] S. V. Manakov, Zh. Eksp. Teor. Fiz. **65**, 505 (1973) [Sov. Phys. JETP **38**, 248 (1974)].
[19] Y. Inoue, J. Plasma Phys. **16**, 439 (1976).
[20] M. V. Tratnik and J. Sipe, Phys. Rev. A **38**, 2011 (1988).
[21] D. N. Christodoulides and R. I. Joseph, Opt. Lett. **13**, 53 (1988).
[22] Y. S. Kivshar, J. Opt. Soc. Am. B **7**, 2204 (1990).
[23] C. Paré and M. Florjanczyk, Phys. Rev. A **41**, 6287 (1990).
[24] T. Ueda and W. K. Kath, Phys. Rev. A **42**, 563 (1990).
[25] J. C. Bhakta, Phys. Rev. E **49**, 5731 (1994).